ON WELL-POSEDNESS AND STABILITY OF THE TWO-FLUID MODEL FOR VERTICAL BUBBLY FLOWS

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Abstract

A well-posed two-fluid model (TFM) system of equations is necessary to obtain results that are physically meaningful. It is known that an incomplete TFM, leads to imaginary roots of the characteristic polynomial, thus rendering the model ill-posed. A common approach to fix this problem has been to introduce sufficient numerical/artificial diffusion to stabilize the model. The disadvantage is that the physical instabilities which are to be predicted by the TFM either disappear or get severely dampened in this process. An alternative is to introduce appropriate physics that stabilize the TFM at short wavelengths while preserving the physical long wavelength instabilities. For instance, in near-horizontal stratified flows the appropriate physical mechanism is surface tension. However, it is not apparent what such a mechanism should be in dispersed bubbly flows.

It was demonstrated by Pauchon and Banerjee [1] that the inclusion of the momentum transfer due to interfacial pressure along with virtual mass force makes the model conditionally well-posed up to a gas volume fraction of 26%. However, in practice one may observe bubbly flows having gas concentrations beyond this limit. Hence it is important to make the behavior of the TFM well-posed for the entire range of gas volume fractions that is physically permissible. In this paper, the mechanism of bubble collisions is considered to make the TFM well-posed. Furthermore, it is also shown that the introduction of collision force model is necessary to get the right void wave propagation velocities. Comparisons are made with the data of Kytomaa and Brennan [2]. Finally, it is demonstrated with the CFD calculations on a bubble column that for the case of vertical bubbly flows, if the appropriate physical mechanisms are considered, the TFM can be made well-posed.

Keywords: well-posedness, collision, linear stability analysis, two-fluid model,

1. Introduction

The two-fluid model (TFM) of Ishii [3] comprises of a set of governing equations for each of the constituent phases. The TFM is very elaborate and it may be applied to any of the two-phase flow regimes. Due to its complex nature, it is a common practice to include terms that are straightforward to implement, and for which closure relations exist. As a result, there is always a possibility of missing essential physics while implementing the TFM in a code. This may result in the model being ill-posed because of incompleteness.

To overcome the issue of ill-posedness, the TFM is regularized by adding artificial terms to the governing equations or by the use of a coarse grid or first order numerical schemes which introduce sufficient numerical damping. The disadvantage of using such a method is that, if the system is over-damped, the two-phase flow instabilities may also be eliminated along with the high frequency oscillations arising from the ill-posedness of the TFM. The approach used here is to regularize TFM on a physical basis so that the inherent dynamics of the flow are preserved. Previously, Pauchon and Banerjee [1] and Park et al. [4] have shown that the bubbly flow TFM can be made conditionally well-posed (up to 26% void fraction) by considering the contributions from the interfacial pressure difference and virtual mass force.

It has been concluded by researchers including Haley et al. [5] and Park et al. [4] that the transition of the eigenvalues from being real-valued to complex-valued is associated with the flow regime transition. However, in some practical applications involving bubbly flows, it may be observed that local void fraction may exceed 26% where the TFM still needs to be well-posed which cannot be achieved with the existing closure models. In the present work the physical mechanism considered to regularize the TFM beyond this limit is a pressure force arising bubble collisions. In the framework of particle flows, there exist quite a few models to account for inter-particle collisions including Ogawa et al. [6], Lun et al. [7], and Boelle et al. [8]. Most of them are derived from the collision kernel which occurs on the right hand side of the Boltzmann transport equation. The model of Alajbegovic et al. [9] is used here, which has been benchmarked with particle flows having similar sizes comparable to bubbles in air-water two-phase bubbly flows. In the present research, it is shown that the collision force makes the TFM unconditionally well-posed. In addition, it is shown that the proposed collision force is capable of giving a reasonable estimation for the material wave velocities. Finally a CFD application illustrates the advantage of the approach.

2. Two-fluid model

The TFM consists of a set of governing equations for the constituent phases treating them as inter-penetrating continua. The current focus is on the hydrodynamic interaction between the constituent phases, and hence the energy equations will not be considered. Two-phase flows are observed to have quite a few flow instabilities which arise due to difference in density and/or relative velocity between the two phases. Since the TFM allows each of the phases to have its own velocity field, it has the capability to model flow dynamics driven by the relative velocity between the phases. It is an important tool in analyzing transient phenomena such as sudden mixing of phases or flow regime transition where the constituent phases are weakly coupled. In comparison, in the drift flux model introduced by Zuber and Findlay [10], the velocities of the phases are related by the drift flux correlations depending on the flow regime and it cannot resolve local instabilities. However the drift flux model is often considered reliable for flows that are strongly coupled and it can represent global instabilities, e.g., flow excursions and density wave oscillations.

For the current research, the adiabatic TFM applied to vertical bubbly two-phase flows is considered for which the continuity and momentum equations described for each phase are given by (Ishii [3]),

$$\frac{\partial}{\partial t}\alpha_k\rho_k + \nabla.\alpha_k\rho_k\bar{u}_k = \Gamma_k \tag{1}$$

$$\frac{\partial}{\partial t}\alpha_k\rho_k\bar{u}_k + \nabla.\alpha_k\rho_k\bar{u}_k\bar{u}_k = -\alpha_k\nabla p_k + \nabla.\alpha_k(\bar{\bar{\tau}}_k + \bar{\bar{\tau}}_k^T) + \alpha_k\rho_k g + M_{ki} - (p_{ki} - p_k)\nabla\alpha_k + (\bar{\bar{\tau}}_{ki} - \bar{\bar{\tau}}_k).\nabla\alpha_k + \bar{u}_{ki}\Gamma_k \tag{2}$$

where, k = 1 for liquid and k = 2 for gas phase. $\alpha_k, \rho_k, \bar{u}_k$ are the void fraction, density and velocity vector corresponding to phase k. Since the research presented here is restricted to the case of adiabatic flows, the interface mass transfer rate is zero, $\Gamma_k = 0$. Similarly, the momentum transfer due to mass transfer is also neglected. The term M_{ki} represents the averaged contribution from the net momentum transfer occurring at the interface between the two phases. The interfacial momentum transfer term M_{ki} is given by,

$$M_{ki} = M_{ki}^D + M_{ki}^L + M_{ki}^W + M_{ki}^{VM} \tag{3}$$

The terms from left to right represent contributions from drag, lift, wall, and virtual mass forces respectively. Basset force is neglected due to the complexity involved in implementing the term. Table 1 summarizes the models used in the current analysis.

Table 1 : Summary of interfacial momentum transfer models

Force	Model
Drag	Ishii-Zuber [11]
Lift	Auton [12]
Wall	Antal [13]
Virtual mass	Drew and Lahey [14]

For linear stability analysis, the 1-D form of the governing equations are used that are similar to Lopez de Bertodano et al. [15]. The system of equations is given by,

$$\frac{\partial\alpha_1}{\partial t} + \frac{\partial}{\partial x}\alpha_1 u_1 = 0 \tag{4}$$

$$\frac{\partial\alpha_2}{\partial t} + \frac{\partial}{\partial x}\alpha_2 u_2 = 0 \tag{5}$$

$$\rho_1\alpha_1\left(\frac{\partial u_1}{\partial t}+u_1\frac{\partial u_1}{\partial x}\right)$$
$$=-\alpha_1\frac{\partial p_1}{\partial x}+\frac{\partial}{\partial x}\alpha_1(\tau_{kzz}+\tau_{kzz}^T)-\alpha_1\rho_1 g_x-M_{2i}^{VM}-M_{2i}^{D}$$
$$-(p_{ki}-p_k)\frac{\partial \alpha_1}{\partial x} \tag{6}$$

$$\rho_2\alpha_2\left(\frac{\partial u_2}{\partial t}+u_2\frac{\partial u_2}{\partial x}\right)=-\alpha_2\frac{\partial p_{1i}}{\partial x}-\alpha_2\rho_2 g_x+M_{2i}^{VM}+M_{2i}^{D} \tag{7}$$

The momentum transfer contributions due to drag is given by,

$$M_{2i}^{D}=-\frac{3}{4}\alpha_2\rho_1\frac{C_D}{D_b}|u_r|u_r \tag{8}$$

The drag coefficient varies depending on the type of flow regime being analyzed. For air-water turbulent flows, the drag correlation of Ishii and Zuber [11] for a multi-particle system is used,

$$C_D=\frac{2}{3}D_b\sqrt{\frac{g\Delta\rho}{\sigma}}\left(\frac{1+17.67f(\alpha_2)^{6/7}}{18.67f(\alpha_2)}\right)^2 \tag{9}$$

where,

$$f(\alpha_2)=(1-\alpha_2)^{1.5} \tag{10}$$

The momentum transfer due to virtual mass force is given by Drew and Lahey [14] as,

$$M_{2i}^{VM}=\alpha_2\rho_1 C_{VM}\left(\frac{D_1u_1}{Dt}-\frac{D_2u_2}{Dt}\right) \tag{11}$$

For the case of potential flow around a sphere, C_{VM} = 0.5. For deformed bubbles, analytical solutions can be obtained for C_{VM} as shown by Lamb [16].

In the present study, the interfacial pressure force is also considered. The difference between the interfacial pressure and the continuous phase pressure can be obtained using Bernoulli's princple as shown by Stuhmiller [17]. It is given by,

$$p_{1i}-p_1=-C_p\rho_1|u_r|^2 \tag{12}$$

The coefficient C_p = 0.25 is used which is obtained for the case of potential flow by averaging over the surface of the sphere. Drew and Passman [18] conclude that for dispersed flow regime, the pressure in the dispersed phase is almost equal to the pressure at the interface, and hence, $p_{2i}\approx p_2$. The effect of the interfacial pressure difference term on the mathematical behaviour of the TFM will be shown with the help of linear stability analysis.

3. Characteristics

The mathematical behaviour of the TFM can be studied by performing a characteristic (eigenvalue) analysis. As mentioned before, depending on the nature of the characteristic roots, the system can be classified as well-posed or ill-posed. Pauchon and Banerjee [1] were among the first to apply this idea to analyze the TFM for vertical bubbly flows. The system of equations, Eq. (3) – Eq. (6), is recast into the following form,

$$A\frac{\partial}{\partial t}\underline{\phi} + B\frac{\partial}{\partial x}\underline{\phi} + D\frac{\partial^2}{\partial x^2}\underline{\phi} + F = 0 \tag{13}$$

where, $\underline{\phi} = [\alpha_2 \;\, u_2 \, u_1 \, p\,]^T$. As shown by Pauchon and Banerjee [1], the evolution of the solution from an initial condition can be understood by solving,

$$\mathrm{Det}(B - \lambda A) = 0 \tag{14}$$

This results in 4 eigenvalues, 2 of which correspond to acoustic wave speeds and the other 2 to the material wave speeds. The acoustic wave speeds, when considered, are always real for subsonic conditions. It is the material wave speeds which determine the well-posedness of the TFM under consideration. Pauchon and Banerjee [1] define the parameter λ^* as,

$$\lambda^* = \frac{\lambda - u_1}{u_2 - u_1} \tag{15}$$

so that the dimensionless eigenvalue depends only onα_2. In addition, it is shown by Pauchon and Banerjee [1] that it is along λ^* given by Eq. (15) that the quantity conserved can be closely approximated by α_2. The parameters used for the stability analysis are listed in Table 2.

Table 2: Parameters for characteristics analysis

Parameter	Value
ρ_1	1000 kg/m^3
ρ_2	1 kg/m^3
u_1	0 m/s
u_2	0.2 m/s
ν_1	10^{-6} m^2/s
ν_2	1.56 x 10^{-5} m^2/s

As seen in Fig. 1, the TFM for vertical bubbly flows with the interfacial pressure and the virtual mass terms where, $C_p = 0.25, C_{VM} = 0.5$ is well-posed up to $\alpha_2 = 0.26$. This result is in accordance with the work of Pauchon and Banerjee [1], and Park et al. [4]. Physically, it is possible to have bubbly two-phase flows with $\alpha_2 > 0.26$, where the TFM should ideally be well-posed. Even for the case of low superficial gas velocities it can be seen that the void fraction is higher locally, for instance near the entrance region where the spargers are located. It is important that the TFM be well-posed for such cases.

4. Collision

To overcome the issue of conditional well-posedness, the mechanism of collision is considered. It is observed experimentally that in regions of higher void fractions, the bubbles have a higher tendency to collide, being closely packed. It is therefore important to consider a collision mechanism to make the TFM more complete. Although collision model derived assuming hard sphere dynamics and instantaneous, binary collisions is far from ideal for bubble-bubble interactions, it is currently the only available model. The present study focuses on the impact that such a model can have on the well-posedness of bubbly two-phase flows and not in deriving a more physical model. In the field of fluid-particle flows, there exist several models to account for this effect. Chapman and Cowling [19] were among the first to derive an expression for the

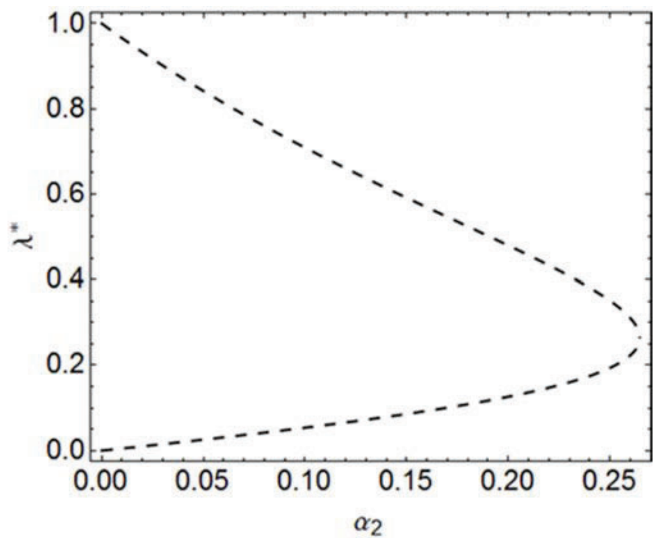

Figure 1: Characteristics with interfacial pressure and virtual mass

collision force starting from the Enskog kinetic equation. A similar approach was adopted by Ogawa et al. [6], Lun et al. [7], and Boelle et al. [8]. In the present research, the collision force model of Alajbegovic et al. [9] is considered which is used for CFD calculations of vertical two-phase flows having density ratio and particle size similar to that of air-water bubbly flows. The expression is given by,

$$M^{coll} = -\nabla.\left[(\rho_2 + \rho_1 C_{VM}) g(\alpha_2)\alpha_2^2\left(2\overline{u_2' u_2'} + \overline{u_2'.u_2'}I\right)\right] \quad (16)$$

which uses the radial distribution function, $g(\alpha_2)$.Here the model proposed by Carnahan and Starling [20] is used,

$$g(\alpha_2) = \frac{2-\alpha_2}{2(1-\alpha_2)^3} \quad (17)$$

which increases considerably with an increase in α_2. It should be noted the contribution from the bubble-bubble collisions is to be considered only in the momentum equation for the gas phase, and not as an action-reaction force pair between the gas and liquid phases.

Eq. (16) is adapted for the case of air-water bubbly flows so that it could be used for linear stability analysis, as well as for numerical calculations. Assuming the bubbles to be in turbulence equilibrium with the continuous phase, the stress tensors in the two phases can be related by,

$$\overline{u_2'u_2'} \approx \overline{u_1'u_1'} \tag{18}$$

The bubble induced component of the stress tensor in the liquid phase given by Lopez de Bertodano et al. [21] is,

$$\overline{u_1'u_1'} = \begin{bmatrix} 4/5 & 0 & 0 \\ 0 & 3/5 & 0 \\ 0 & 0 & 3/5 \end{bmatrix} \frac{1}{2} C_{VM} \alpha_2 |u_r|^2 \tag{19}$$

The particle stress tensor $\overline{u_2'u_2'}$ is assumed to be isotropic. Hence, Eq. (15) can be reduced to,

$$M^{coll} = -C_{coll} \rho_1 C_{VM}^2 \nabla . \left[g(\alpha_2) \alpha_2^3 |u_r|^2 I \right] \tag{20}$$

For the case of linear stability analysis, Eq. (21) in 1-D form can be rewritten as,

$$M^{coll} = -C_{coll} \rho_1 C_{VM}^2 \left(\left(3\alpha_2^2 u_r^2 g(\alpha_2) + \alpha_2^3 u_r^2 \frac{dg(\alpha_2)}{d\alpha_2} \right) \frac{\partial \alpha_2}{\partial x} + 2\alpha_2^3 u_r g(\alpha_2) \frac{\partial u_r}{\partial x} \right) \tag{21}$$

where, $C_{coll} = 1.0$ is used for simplicity. When Eq. (13) is evaluated after adding the collision term, it can be seen that the roots associated with the material waves are real for the entire void fraction range. The TFM is thus made unconditionally well-posed by regularizing the model with the addition of missing physics. In addition, it can be observed (Fig. 3) that the prediction of the material wave velocities from the TFM developed in the present study is in reasonable agreement with the data of Kytomaa and Brennan [2].

5. CFD Analysis

The TFM developed in the present study is extended for 3-D CFD analysis. The Eulerian TFM in Ansys Fluent 15.0 is used to perform the CFD calculations. The experiment of Reddy Vanga [22] is chosen, where air-water two-phase bubbly flow experiments were conducted to study the buoyancy driven plume instability. The computational domain consists of a rectangular cross-section 0.1 m by 0.02 m having a height of 0.3 m. The gas superficial velocity at the inlet was set at 2 mm/s. The mesh sizes chosen are Δx = 5 mm, and 1.25 mm. The respective time step sizes are Δt = 8 ms, and 2 ms. The QUICK scheme is used for spatial discretization, and a second-order implicit scheme is used for time marching. The higher-order numerics ensure that the effect of numerical diffusion is minimized.

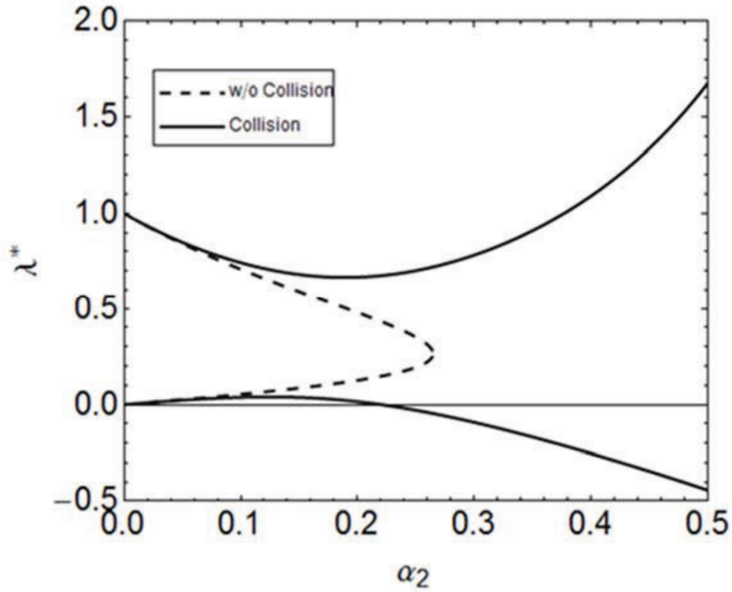

Figure 2: Characteristics with collision

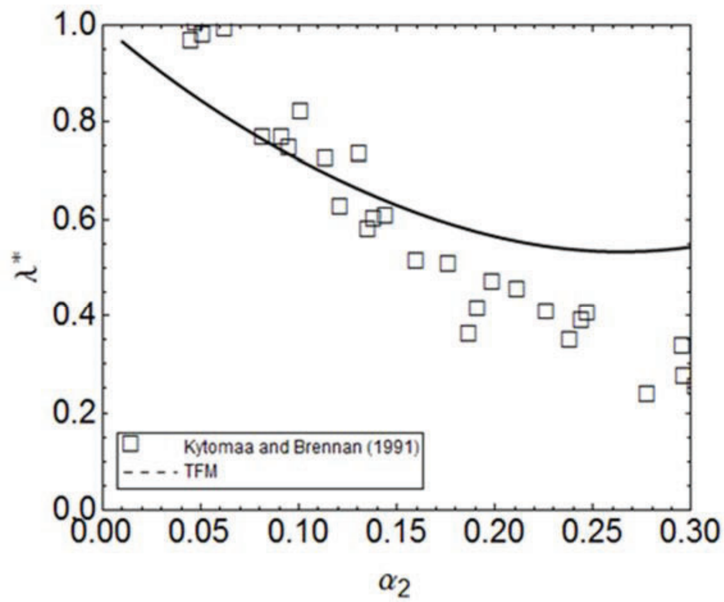

Figure 3: Comparison with the data of Kytomaa and Brennan [2]

Initially, the default CFD TFM is used without the inclusion of interfacial pressure or collision mechanisms. It is observed that the CFD calculations with 5 mm mesh captures the plume dynamics as seen in Fig. 4, which shows instantaneous void fraction distributions at three different instances. When the mesh is refined, non-physical high frequency void fraction distributions start to appear as seen in Fig. 5. This is an indicator of ill-posedness in the TFM used for CFD calculations. When the interfacial pressure and collision terms are added, the TFM becomes well-posed. It is shown in Fig. 5 that the solution is free from high frequency void fraction waves when the regularized TFM is used. Thus it is shown that the linear stability analysis for 1-D TFM can be extended to 3-D TFM to resolve the issue of ill-posedness.

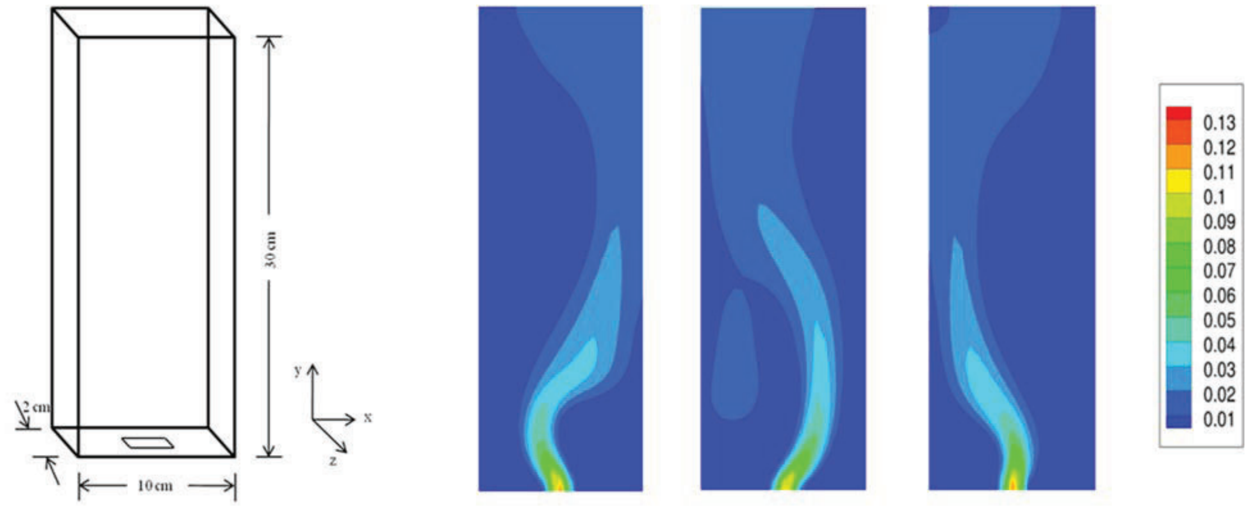

Figure 4: Instantaneous void fraction distributions for Δx = 5 mm

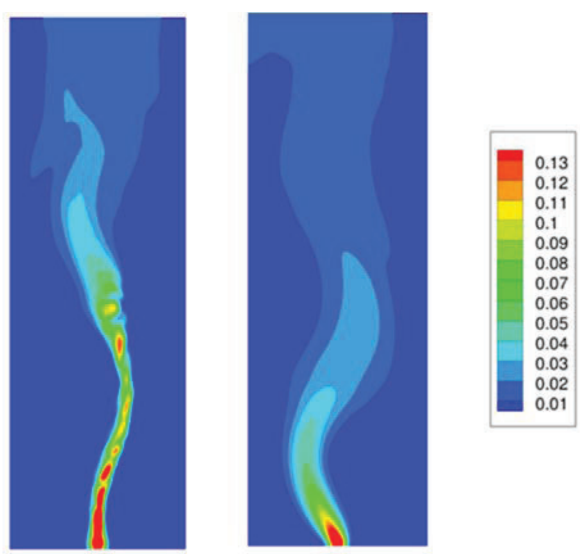

Figure 5: Results with (left) ill-posed, (right) well-posed TFM for Δx = 1.25 mm

6. Conclusions

A well-posed TFM for bubbly flow is developed in the present study based on a mechanistic approach. Physical regularization is achieved by adding the momentum transfer from bubble collisions. The effect collisions on stability is demonstrated with the help of linear stability analysis. Previously, it was shown by Pauchon and Banerjee [1], Haley et al. [5], and Park et al. [4] that the TFM is conditionally well-posed with the cut off void fraction being dependent on the coefficients of the closure relations for virtual mass and interfacial pressure. With the addition of collision mechanism, it is seen that the TFM becomes well-posed for the entire void fraction range of practical interest. This is a significant improvement considering the fact that the bubbly flows can prevail even at instances where the local void fraction goes beyond $\alpha_2 = 0.26$. Additionally, it is observed that the material wave velocity obtained with the TFM developed in the present study agrees reasonably well with the data of Kytomaa and Brennan [2]. Finally, it is shown that the analysis with 1-D TFM can be extended to a CFD application and that convergence study becomes feasible.

References

[1] C. Pauchon and S. Banerjee, "Interphase Momentum Interaction Effects in the Averaged Multifield Model, Part 1: Void Propagation in Bubbly Flows," *International Journal of Multiphase Flow,* vol. 12, pp. 559-573, 1986.

[2] H. K. Kytomaa and C. E. Brennen, "Small Amplitude Kinematic Wave Propagation in Two-component Media," *International Journal of Multiphase Flow,* vol. 17, no. 1, pp. 13-26, 1991.

[3] M. Ishii, Thermo-fluid Dynamic Theory of Two-phase Flow, Paris: Eyrolles, 1975.

[4] J. W. Park, D. A. Drew and R. T. Lahey, "The Analysis of Void Wave Propagation in Adiabatic in Adiabatic Monodispersed Bubbly Two-phase Flows Using an Ensemble-averaged Two-fluid Model," *International Journal of Multiphase Flow,* vol. 24, pp. 1205-1244, 1998.

[5] T. Haley, D. Drew and R. Lahey, "An analysis of the EigenValues of Bubbly Two-phase Flows," *Chemical Engineering Communications,* vol. 106, pp. 93-117, 1991.

[6] S. Ogawa, A. Umemura and N. Oshima, "On the Equations of Fully Fluidized Granular Materials," *Journal of Applied Mathematics and Physics,* vol. 31, pp. 483-493, 1980.

[7] C. K. Lun, S. B. Savage, D. J. Jeffrey and N. Chepurniy, "Kinetic Theories for Granular Flow: Inelastic Particles in Couette Flow and Slightly Inelastic Particles in a General Flowfield," *Journal of Fluid Mechanics,* vol. 140, pp. 223-256, 1984.

[8] A. Boelle, G. Balzer and O. Simonin, "Second-order Prediction of the Particle-phase Stress Tensor of Inelastic Spheres in Simple Shear Dense Suspensions," in *Gas Particle Flows, ASME FED*, 1995.

[9] A. Alajbegovic, D. A. Drew and R. T. Lahey Jr., "An Analysis of Phase Distribution and Turbulence in Dispersed Particle/liquid Flows," *Chemical Engineering Communications,* vol. 174, pp. 85-133, 1999.

[10] N. Zuber and J. A. Findlay, "Average Volumetric Concentration in Two-phase Flow Systems," *Journal of Heat Transfer,* vol. 87, pp. 453-468, 1965.

[11] M. Ishii and N. Zuber, "Drag Coefficient and Relative Velocity in Bubbly, Droplet or Particulate Flows," *AIChE,* vol. 25, pp. 843-855, 1979.

[12] T. R. Auton, "The Lift Force on a Spherical Body in a Rotational Flow," *Journal of Fluid Mechanics,* vol. 183, pp. 199-218, 1987.

[13] S. P. Antal, R. T. Lahey and J. E. Flaherty, “Analysis of Phase distribution in Fully Developed Laminar Bubbly Two-phase Flow,” *International Journal of Multiphae Flow,* vol. 17, pp. 635-652, 1991.

[14] D. A. Drew and R. T. Lahey Jr., “The Virtual Mass and Lift Force on a Sphere in Rotating and Straining Inviscid Fluid,” *International Journal of Multiphase Flow,* vol. 13, no. 1, pp. 113-121, 1987.

[15] M. Lopez de Bertodano, W. Fullmer and A. Vaidheeswaran, “One-dimensional Two-equation Two-fluid Model Stability,” *Multiphase Science and Technology,* vol. 25, pp. 133-167, 2013.

[16] H. Lamb, Hydrodynamics, New York: Dover, 1932.

[17] J. H. Stuhmiller, “The Influence of Interfacial Pressure Forces on the Character of Two-phase Flow Model Equations,” *International Journal of Multiphase Flow,* vol. 3, pp. 551-560, 1977.

[18] D. A. Drew and S. L. Passman, Theory of Multicomponent Fluids, New York: Springer-Verlag, 1998.

[19] S. Chapman and T. G. Cowling, The Mathematical Theory of Non-uniform Gases, London: Cambridge University Press, 1970.

[20] N. F. Carnahan and K. E. Starling, “Equations of State for Non-attracting Rigid Spheres,” *Journal of Chemical Physics,* vol. 51, pp. 635-636, 1969.

[21] M. Lopez de Bertodano, R. T. Lahey Jr. and O. C. Jones, “Development of a k-epsilon Model for Bubbly Two-phase Flow,” *Journal of Fluids Engineering,* vol. 116, pp. 128-134, 1994.

[22] B. N. Reddy Vanga, *Experimental Investigation and Two-Fluid Model Large Eddy Simulations of Re-circulating Turbulent Flow in Bubble Columns,* West Lafayette, Indiana: Purdue University, 2004.